Homework 7: Solutions to exercises not appearing in Pressley

Math 120A

- (4.5.3) Recall we have an atlas for the sphere S^2 consisting of $\sigma_1 : U_1 \to S^2$ and $\sigma_2 : U_2 \to S^2$ where $U_1 = U_2 = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (0, 2\pi)$ with

$$\sigma_1(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$$
$$\sigma_2(\theta, \phi) = (-\cos\theta\cos\phi, -\sin\theta, -\cos\theta\sin\phi)$$

For σ_1, we have the partial derivatives

$$(\sigma_1)_\theta = (-\sin\theta\cos\phi, -\sin\theta\sin\phi, \cos\theta)$$
$$(\sigma_1)_\phi = (-\cos\theta\sin\phi, \cos\theta\cos\phi, 0)$$

with cross product $(\sigma_1)_\theta \times (\sigma_1)_\phi = (-\cos^2\theta\cos\phi, -\cos^2\theta\sin\phi, -\cos\theta\sin\theta) = -\cos\theta\sigma(\theta, \phi)$. Since $\cos\theta$ is a positive number on $(-\frac{\pi}{2}, \frac{\pi}{2})$, this vector points in the opposite direction as $\sigma(\theta, \phi)$, and in particular points inward toward the origin on the unit sphere. Therefore so does the standard unit normal. The computation for σ_2 is extremely similar.

- Question 3. For the first part, suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at x. Then let

$$\epsilon(h) = \frac{|h|}{h}\left[\frac{f(x+h) - f(x)}{h}\right]$$

Then $\epsilon(h) \to 0$ as $h \to 0$, and we have

$$f(x+h) = f(x) + f'(x)(h) + |h|\epsilon(h)$$

Therefore f satisfies the new definition with $T_{\mathbf{p}}(h) = f'(x)h$.

For the second part, let $f : \mathbb{R}^n \to R^m$ have component functions $f_i(x_1, \cdots, x_n)$ for $i = 1, \cdots m$. Assume f is differentiable at $\mathbf{p}$ and let $\mathbf{v} = h\mathbf{e}_j$ for some $h > 0$ and $\mathbf{e}_j$ the jth standard basis vector. Then we have

$$f(\mathbf{p} + h\mathbf{e}_j) = f(\mathbf{p}) + T_{\mathbf{p}}(h\mathbf{e}_j) + |h|\epsilon(h\mathbf{e}_j)$$

Rearranging and recalling that $T_{\mathbf{p}}$ is linear shows that

$$\lim_{h\to 0} \frac{f(\mathbf{p} + h\mathbf{e}_j) - f(\mathbf{p}}{h} = T_{\mathbf{p}}(\mathbf{e}_j)$$

Breaking the right side up into coordinates shows that

$$\begin{aligned}
T_{\mathbf{p}}(\mathbf{e}_j) &= \left(\lim_{h \to 0} (\frac{f_1(\mathbf{p} + h\mathbf{e}_j) - f_1(\mathbf{p}}{h}, \cdots, \frac{f_m(\mathbf{p} + h\mathbf{e}_j) - f_m(\mathbf{p}}{h} \right) \\
&= \left(\frac{\partial f_1}{\partial x_j}(\mathbf{p}), \cdots, \frac{\partial f_m}{\partial x_j}(\mathbf{p}) \right)
\end{aligned}$$

Since $T_{\mathbf{p}}(\mathbf{e}_j)$ is the jth column of the matrix of $T_{\mathbf{p}}$ with respect to the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, we conclude this matrix is the Jacobian of f.

For the last part, we see that the function in the last problem of HW 1 has partial derivatives (and therefore a Jacobian) at $(0,0)$, but is not continuous at $(0,0)$, and therefore by the proposition not differentiable at $(0,0)$ either.